Melding priority queues *

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Abstract

We show that any priority queue data structure that supports *insert*, *delete*, and *find-min* operations in $pq(n)$ amortized time, when n is an upper bound on the number of elements in the priority queue, can be converted into a priority queue data structure that also supports fast *meld* operations with essentially no increase in the amortized cost of the other operations. More specifically, the new data structure supports *insert*, *meld* and *find-min* operations in $O(1)$ amortized time, and *delete* operations in $O(pq(n) + \alpha(n, n))$ amortized time, where $\alpha(m, n)$ is a functional inverse of the Ackermann function. The construction is very simple. The meldable priority queues are obtained by placing a non-meldable priority queue at each node of a union-find data structure. We also show that when all keys are integers in the range $[1, N]$, we can replace n in the bound stated above by $\min\{n, N\}$.

Applying this result to non-meldable priority queue data structures obtained recently by Thorup, and by Han and Thorup, we obtain meldable RAM priority queues with $O(\log\log n)$ amortized cost per operation, or $O(\sqrt{\log\log n})$ expected amortized cost per operation, respectively. As a by-product, we obtain improved algorithms for the minimum directed spanning tree problem in graphs with integer edge weights: A deterministic $O(m \log\log n)$ time algorithm and a randomized $O(m\sqrt{\log\log n})$ time algorithm. These bounds improve, for sparse enough graphs, on the $O(m + n \log n)$ running time of an algorithm by Gabow, Galil, Spencer and Tarjan that works for arbitrary edge weights.

Key Words: Priority queues, heaps, union-find, word RAM model, optimum branchings, minimum directed spanning trees.

1 Introduction

Priority queues are basic data structures used by many algorithms. The most basic operations, supported by all priority queues, are *insert*, which inserts an element with an associated key into the priority queue, and *extract-min*, which returns the element with the smallest key currently in the queue, and deletes it. These two operations can be used, for example, to sort n elements by performing n *insert* operations followed by n *extract-min* operations. Most priority queues also support a *delete* operation, that deletes a given element, not necessarily with the minimum key, from the queue, and *find-min*, which finds, but does not delete, an element with minimum key.

The efficient implementation of several algorithms, such as the algorithm of Edmonds [12] for computing optimum branching and minimum directed spanning trees, require the maintenance of a collection of priority queues. In addition to the standard operations performed on individual priority queues in this collection, these algorithms also need, quite often, to *meld*, or unite, two priority queues from this collection. This provides a strong motivation for studying *meldable* priority queues.

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Fibonacci heaps, developed by Fredman and Tarjan [13], are very elegant and efficient meldable priority queues. They support *delete* operations in $O(\log n)$ *amortized* time, where n is the size of the priority queue from which an element is deleted, and all other operations, including *meld*, in constant amortized time. For a general discussion of amortized time bounds, see [30]. Most importantly, amortized time bounds suffice when priority queues are used as subroutines by algorithms for static problems, like the algorithm of Edmonds mentioned above for finding optimal branchings.

Fibonacci heaps are particularly famed for supporting a special *decrease-key* operation in constant time. While a fast *decrease-key* operation in important in the implementation of Dijkstra's single source shortest paths algorithm [10], we are not aware of any application that requires both *meld* and *decrease-key* operations. Thus, we will not consider *decrease-key* operations any further in this paper.

While $O(\log n)$ is the best *delete* time possible in the comparison model, where keys can only be compared, much better time bounds can be obtained in the word RAM model of computation, as was first demonstrated by Fredman and Willard [14, 15]. In this model each key is assumed to be an integer that fits into a single word of memory. Each word of memory is assumed to contain $w \geq \log n$ bits. The model allows the use of keys for random access to memory. The set of basic word operations that can be performed in constant time are the standard word operations available in typical programming languages (e.g., C): addition, multiplication, bit-wise and/or operations, shifts, and their like. The word RAM model is required for hashing [11] and for classic radix sort algorithm which dates back to at least 1929 [9].

In the word RAM model, Thorup [32] obtained a general equivalence between priority queues and sorting. More specifically, he showed that if up to n elements can be sorted in $O(n \cdot s(n))$ time, then the basic priority queue can be implemented with *delete* in $O(s(n))$ time and *insert* and *find-min* in constant time. The $O(n \log\log n)$ sorting algorithm of Han [17] gives *delete* in $O(\log\log n)$ time. Similarly, the randomized $O(n\sqrt{\log\log n})$ expected time sorting algorithm of Han and Thorup [18] gives *delete* in $O(\sqrt{\log\log n})$ time.

Alstrup *et al.* [1] describe a simple transformation that show that any priority queue data structure that supports the basic priority queue operations in $O(pq(n))$ time can be converted into a data structure that supports *insert* and *find-min* operations is $O(1)$ amortized time and *delete* operations is $O(pq(n))$ amortized time.

1.1 Adding a meld operation

The fast priority queues mentioned above, with the $O(\log\log n)$ time or $O(\sqrt{\log\log n})$ expected time per operation, do *not* support meld operations. Our main result is a general word RAM transformation that takes these priority queues, or any other priority queue data structure, and produces new priority queue data structures that support *meld* operations in constant amortized time, without increasing the amortized cost of the other priority queue operations! More specifically, we show that any priority queue data structure that supports *insert*, *delete*, and *find-min* operations in $pq(n)$ amortized time, where n is the number of elements in the priority queue, can be converted into a priority queue data structure that supports *insert*, *meld* and *find-min* operations in constant amortized time, and *delete* operations in $O(pq(n) + \alpha(n, n))$ amortized time, where $\alpha(m, n)$ is a functional inverse of the Ackermann function (see [27]). The function $\alpha(n, n)$ is an extremely slow growing function. In particular, $\alpha(n, n) = o(\log^* n)$.

Applying our generic transformation to the non-meldable priority queues mentioned above, we get word RAM meldable priority queues that support *insert*, *meld* and *find-min* operations in $O(1)$ amortized time, and *delete* operations in $O(\log\log n)$ amortized time, or $O(\sqrt{\log\log n})$ expected amortized time.

In combination with Thorup's [32] reduction from non-meldable priority queues to sorting, we get that if there is a sorting algorithm that can sort up to n elements in $O(n \cdot s(n))$ time, then there are also meldable priority queues supporting *insert*, *meld* and *find-min* operations in $O(1)$ amortized time, and *delete* operations in $O(s(n) + \alpha(n, n))$ amortized time.

1.2 Avoiding multiplication

The transformation that yields meldable priority queues with an amortized *delete* time of $O(pq(n) + \alpha(n, n))$ uses the *atomic heaps* of Fredman and Willard [15]. These, in turn, use multiplication, which is not an AC^0 operation.

We also describe, however, a simpler combinatorial transformation that uses non-meldable priority queues as black boxes and otherwise uses only standard AC^0 word operations, that is, comparison, addition, shifts, and bit-wise Boolean operations. The transformation only accesses the keys via the non-meldable priority queues, so if these are comparison based, then so are the resulting meldable priority queues. With this simpler transformation, if the basic black box priority queues support all operations in $O(pq(n))$ amortized time, then the meldable priority queue obtained supports *insert*, *meld* and *find-min* operations in constant amortized time, and *delete* operations in $O(pq(n)\,\alpha(n, n/pq(n)))$ amortized time. We note that $\alpha(n, n/pq(n))$ is *constant* for, e.g., $pq(n) = \Omega(\log^* n)$.

The above mentioned transformation of Thorup [32] from sorting to priority queues uses multiplication, but it also comes in a simpler combinatorial variant that can be implemented on a pointer machine using only standard AC^0 word operations. Given a sorting algorithm sorting up to n keys in $O(s(n))$ time per key, he gets a basic priority queue supporting all operations in $O(pq_s(n))$ time, where the function $pq_s(n)$ satisfies the recurrence relation $pq_s(n) = O(pq_s(\log^2 n) + s(n))$. We note that $pq_s(n) = O(s(n))$ if $s(n) = O(\log^{(i)} n)$, for any $i \geq 1$, i.e., if $s(n)$ is bounded by any iterated log-function. However, $pq_s(n) = \Omega(\log^* n)$ even if $s(n) = O(1)$, so in combination with our own combinatorial transformation, we get a meldable priority queue with an amortized *delete* time of $O(pq_s(n)\,\alpha(n, n/pq_s(n))) = O(pq_s(n))$ and all other operations in amortized constant time.

Restricting ourselves to a word RAM with only standard AC^0 operations, we can now use the $O(n \log\log n)$ expected time AC^0 sorting algorithm of Thorup [33] to get meldable priority queues with *delete* in $O(\log\log n)$ expected amortized time. Alternatively, for any $\varepsilon > 0$, we can use the $O(n(\log\log n)^{1+\varepsilon})$ time deterministic AC^0 sorting algorithm of Han and Thorup [18] to get an amortized *delete* time of $O((\log\log n)^{1+\varepsilon})$.

1.3 Improvement for smaller integer keys

In some cases, the keys are small compared to n. We also describe a second independent transformation that shows that a meldable priority queue data structure that supports all operations in $pq(n)$ time, where n is a bound on the number of elements in the priority queues, can be used to construct a priority queue data structure that supports *insert*, *meld* and *find-min* operations in $O(1)$ amortized time, and *delete* operations in $O(pq(\min\{N, n\}))$ amortized time, when all keys are integers in the range $[1, N]$.

1.4 Optimum branchings and minimum directed spanning trees

We will now describe the main context in which our fast meldable priority queues may be used.

The problem of finding a minimum (or maximum) spanning tree in an *undirected* graph is an extremely well studied problem. Chazelle [6] obtained a deterministic $O(m\,\alpha(m, n))$ time algorithm for the problem. An asymptotically optimal algorithm for finding minimum spanning trees, with an unknown running time, was given by Pettie and Ramachandran [25]. Karger *et al.* [20] obtained a randomized algorithm that runs, with very high probability, in $O(m + n)$ time. All these algorithms are comparison based and can handle arbitrary real edge weights. Fredman and Willard [15] obtained a deterministic $O(m + n)$ time algorithm for the problem in the word RAM model.

The directed version of the minimum spanning tree problem has received much less attention in recent years. This version comes in different variants. We are either given a root r and asked to find a directed spanning tree of minimum weight rooted at r, or we are asked to find a directed spanning tree of minimum weight rooted at an arbitrary vertex. (It is assumed, in both cases, that the desired directed spanning trees do exist.) A very closely related problem is the problem of finding an *optimum branching*, i.e., a branching of maximum total weight. A branching B in a directed graph is a collection of edges that satisfies the following two properties: (i) B does not contain a cycle; (ii) No two edges of B are directed into the same vertex. It is not difficult to show that these three versions are essentially equivalent. We refer to the three of them collectively as the *minimum directed spanning tree (MDST)* problem. All results stated in this paper apply to all three versions.

Chu and Liu [7], Edmonds [12] and Bock [2] independently obtained an essentially identical polynomial time algorithm for the MDST problem. (In the sequel we refer to this paper, somewhat unfairly, as Edmonds' algorithm.) Karp [21] gave a simple formulation of the algorithm and a direct combinatorial proof of its correctness. All subsequent results, including ours, are just more efficient implementations of variants of this algorithm using improved *meldable* priority queue data structures.

Tarjan [28] (see also Camerini *et al.* [5]) describes a natural way of implementing the algorithm of Edmonds using a meldable priority queue data structure. The complexity of the algorithm is dominated by the cost of performing a sequence of at most $O(m)$ *insert* operations, $O(m)$ *extract-min* operations, $O(n)$ *meld* operations, and $O(n)$ *add* operations. An *add* operations adds a given constant to the keys of all the elements contained in a certain priority queue. Adding an *add* operation to existing priority queue data structures is not a difficult task. (For the details, see [28].) In particular, it is not difficult to augment the priority queues obtained using our transformations with a constant time *add* operation. Using simple meldable priority queues with $O(\log n)$ time per operation, Tarjan [28] obtains an $O(\min\{m \log n, n^2\})$ algorithm for the problem.

Gabow *et al.* [16] give a more sophisticated algorithm that implements a variant of Edmonds's algorithm using at most $O(n)$ *insert* operations, $O(n)$ *extract-min* operations, $O(n)$ *delete* operations, $O(n)$ *add* operations, and finally $O(m)$ *move* operations. A *move* is a non-standard priority queue operation that moves an element from one priority queue to another. There are no known constructions that support such an operation in constant time, without severely deteriorating the cost of the other operations. Although Fibonacci heaps do not support a general move operation in constant time, Gabow *et al.* [16] show that in the special context of their algorithm, the required move operations can be implemented in constant time. As a result, they obtain an $O(m + n \log n)$ algorithm for the minimum directed spanning tree problem.

Our improved RAM algorithms for the minimum directed spanning tree problem are obtained by plugging in our improved meldable priority queues into Tarjan's [28] implementation of Edmonds' algorithm. We thus obtain a deterministic $O(m \log \log n)$ time algorithm and the randomized $O(m\sqrt{\log \log n})$ expected time algorithm.

We were not able to augment our improved meldable priority queues with a fast *move* operation required by the approach of Gabow *et al.* [16]. Obtaining algorithms with a running time of the form $O(m + nf(n))$, where $f(n) = o(\log n)$, remains a challenging open problem.

1.5 Techniques

Our simple combinatorial transformation is essentially due to van Emde Boas *et al.* [36]. Our main contribution is an improved analysis of this transformation. To construct meldable priority queues using non-meldable ones, we place a non-meldable basic priority queue at each node of a union-find data structure. Suppose that the non-meldable priority queue placed at each node of the union-find data structure supports *delete* operations in $O(pq(n))$ amortized time. A *delete* operation on the meldable priority queue may result in an amortized number of $O(\alpha(n, n))$ *delete* operations on non-meldable priority queues. This gave van Emde Boas *et al.* [36] an amortized meldable delete time of $O(pq(n)\,\alpha(n, n))$. Our improved analysis shows that the meldable *delete* time is actually bounded by $O(pq(n)\,\alpha(n, n/pq(n)))$. The significance of the improvement is that $\alpha(n, n/pq(n))$ is constant for, e.g., $pq(n) = \Omega(\log^* n)$. Hence we do not incur any asymptotic overhead from the transformation using existing non-meldable priority queues.

To get the slightly improved bound of $O(pq(n) + \alpha(n, n))$, we replace small non-meldable priority queues that appear at the bottom of the union-find trees by the *atomic heaps* of Fredman and Willard [15]. An atomic heap has a constant operation time, but it can hold only $O(\log^2 n)$ elements. Atomic heaps use multiplication and Thorup [34] has shown that they cannot be implemented using standard AC^0 operations.

Originally, van Emde Boas *et al.* [36] used the transformation to get a meldable priority queue that supports all operations in $O((\log \log N)\alpha(n, n))$ amortized time. Our improved analysis shows that the amortized time is $O(\log \log N)$. We note, however, that a meldable priority queue with this time bound was already obtained by Bright [3] using a specialized construction. Our contribution is the general transformations giving all the other stated time bounds for meldable priority queues.

1.6 The conference versions

This paper is the combined journal version of [23, 24]. While writing these conference versions we were not aware of the results of [3, 36] and thus presented the transformation of van Emde Boas' *et al.* [36] as our own. The main contribution of this paper is an improved analysis of the transformation, showing that we do not incur any asymptotic loss in the operation time by adding meld operations to existing non-meldable priority queues.

In [24] we also presented some weaker meldable priority queues that support constant time *decrease-key* operation. It appears, however, that better results can be obtained using an entirely different construction related to that of Bright [3]. As mentioned previously, we are not aware of any application that requires both *meld* and *decrease-key*operations. We decided, therefore, not to consider *decrease-key* operations any further in this paper.

1.7 Organization of paper

The rest of this paper is organized as follows. In the next section we review the classical union-find data structure and prove a simple lemma that we shall need about its behavior. As mentioned, the meldable priority queues are obtained by 'planting' a non-meldable priority queue at each node of the union-find data structure. In Section 3 we describe the transformation from non-meldable priority queues into meldable ones. In Section 4 we present our improved analysis of the transformation. In Section 5 we present the slightly improved version of the transformation that employs atomic heaps. In Section 6 we describe an independent simple transformation that allows us to obtain time bounds that depend only on N, the maximum key value, and not on n, the number of elements in the priority queue. We end in Section 7 with some concluding remarks and open problems.

2 The Union-find data structure

A union-find data structure supports the following operations:

make-set(x)	– Create a set that contains the single element x.
union(x, y)	– Unite the sets containing the elements x and y.
find(x)	– Return a representative of the set containing the element x.

A classical, simple, and extremely efficient implementation of a union-find data structure is given in Figure 1. Each element x has a parent pointer $p[x]$ and a number *rank*$[x]$ associated with it. The parent pointers define trees that correspond to the sets maintained by the data structure. The representative element of each set is taken to be the root of the tree containing the elements of the set. To find the representative element of a set, we could simply follow the parent pointers until we get to a root. However, to speed-up future *find* operations, we employ the *path compression* heuristic that makes all the vertices encountered on the way to the root direct children of the root. Unions are implemented using the *union by rank* heuristic: a single element has rank 0. When sets are united, the root of smaller rank is hung on the other. In case of ties, we arbitrarily hang one root on the other and increment the rank of the surviving root by one. Thus, the ranks along any path towards the root of the tree form an increasing sequence, and the rank of an element is an upper bound on the depth of its subtree.

In a seminal paper, Tarjan [27] showed that the time taken by the algorithm of Figure 1 to process an intermixed sequence of m *make-set*, *union* and *find* operations, out of which n are *make-set* operations, is $O(m\,\alpha(m,n))$, where $\alpha(m,n)$ is the extremely slowly growing inverse of Ackermann's function, which we define next.

Ackermann's function is an extremely fast growing function which has many essentially equivalent definitions. One of them, which is taken from Tarjan [27], is:

$$\begin{aligned} A(0,j) &= 2j\,, \quad \text{for } j \geq 1 \\ A(i,1) &= 2\,, \quad \text{for } i \geq 1 \\ A(i,j) &= A(i-1, A(i,j-1))\,, \quad \text{for } i \geq 1,\ j \geq 2 \end{aligned}$$

The inverse Ackermann function $\alpha(m,n)$ is then defined as follows:

$$\alpha(m,n) = \min\{\, i \geq 1 \mid A(i, \lfloor m/n \rfloor) > \log_2 n \,\}\,.$$

A slightly better bound on the number of operations performed by the union-find algorithm was obtained by Tarjan and van Leeuven [31].

Theorem 2.1 (Tarjan and van Leeuven [31]) *The union-find algorithm of Figure 1 processes an intermixed sequence of n make-set operations, up to n link operations, and f find operations in $O(n + f\,\alpha(f+n, n))$ time.*

```
make-set(x) :
p[x] ← x
rank[x] ← 0

union(x, y) :
link(find(x), find(y))

link(x, y) :
if rank[x] > rank[y]
  then p[y] ← x
  else p[x] ← y
       if rank[x] = rank[y]
         then rank[y] ← rank[y] + 1

find(x) :
if p[x] ≠ x
  then p[x] ← find(p[x])
return p[x]
```

Figure 1: The classical union-find data structure

The analysis presented in the Section 4 relies on the following lemma:

Lemma 2.2 *Suppose that an intermixed sequence of n make-set operations, at most n link operations, and at most f find operations are performed on the standard union-find data structure. Then, the number of times the parent pointers of elements of rank k or more are changed is at most $O(\frac{n}{2^k} + f\,\alpha(f + \frac{n}{2^k}, \frac{n}{2^k}))$.*

Proof: A node x is said to be *high* if $rank[x] \geq k$, and *low*, otherwise. Note that all ancestors of high nodes are high and all descendants of low nodes are low. We consider each one of the trees formed by the union-find data structure to be composed of a (possibly empty) *top* part containing high nodes, and a *bottom* part containing low nodes.

The rank of a node cannot decrease, so a high node remains high. We will now prove, by induction on i, that at most $n/2^i$ nodes are ever assigned rank i. The claim trivially holds for $i = 0$. An element of rank i is generated only when one rank $i-1$ root element is hang on another rank $i-1$ root element. These two elements cannot be used to generate other rank i elements as the first is no longer a root and the second now has rank i. By the induction hypothesis, the number of elements that are assigned rank $i-1$ is at most $n/2^{i-1}$. Hence, the number of elements that are assigned rank i is at most $n/2^i$, as required. In particular, we get that the number of high elements is at most $n/2^k$.

We claim that the operations performed on the top parts of the trees correspond to a sequence of at most f *find* operations, and at most $n/2^k$ *link* operations, on the $n/2^k$ high elements. By Theorem 2.1, the number of pointers changed during these operations is at most $O(\frac{n}{2^k} + f\,\alpha(f + \frac{n}{2^k}, \frac{n}{2^k}))$, as required.

The corresponding sequence of operations performed on the top parts of the trees is obtained in the following way. When the rank of an element x becomes k, we add a *make-set*(x) operation to the constructed sequence. When a *link*(x, y) operation is performed, where both x and y are high, we add a *link*(x, y) operation to the constructed sequence. When a *find*(x) operation from the original sequence of operations returns a high element, we add a *find*(x') operation to the constructed sequence, where x' is the first high element on the path from x to its root before the *find*(x) operation. The constructed sequence contains at most $n/2^k$ *make-set* operations, at most $n/2^k$ *link* operations and at most f *find* operations, as required.

It is easy to see that when the union-find data structure is used to process the constructed sequence of operations on the high nodes, the operations performed are exactly those performed on the high elements while processing the original sequence of operations. This completes the proof of the lemma. □

We note in passing that the $O(m\alpha(m, n))$ bound of Tarjan [27] is sufficient for proving Lemma 2.2. This Lemma, in turn, can be used to obtain the slightly tighter bound of Tarjan and van Leeuven [31] (Theorem 2.1 above) on which we rely in Section 5.

3 The transformation

In this section we describe a transformation that combines a non-meldable priority queue data structure with the classical union-find data structure (see [27], [29], or [8]) to produce a *meldable* priority queue data structure with essentially no increase in the amortized operation cost. As mentioned earlier, this transformation was first described in van Emde Boas *et al.* [36], though it is presented there in a somewhat specialized setting. An improved analysis of this transformation appears in the next section.

The transformation $\mathcal{T}$ receives a non-meldable priority queue data structure $\mathcal{P}$ and produces a meldable priority queue data structure $\mathcal{T}(\mathcal{P})$. An element of a priority queue is an identifier x with a key value $key[x]$ associated to it. Later, we shall associate other information, such as parent $p[x]$, with the identifier x. All this information can be accessed in constant time given x.

We assume that the non-meldable data structure $\mathcal{P}$ supports the following operations:

make-pq(x)	– Create and return a priority queue that contains the single element x.
insert(PQ, x)	– Insert the element x into the priority queue PQ.
delete(PQ, x)	– Delete the element x from the priority queue PQ.
find-min(PQ)	– Find and return an element with the smallest key contained in the priority queue PQ.

We can easily add the following operation to the repertoire of the operations supported by this priority queue:

change-key(PQ, x, k) – Change the key of element x in PQ to k.

This is done by deleting the element x from the priority queue PQ, changing its key by setting $key[x] \leftarrow k$, and then reinserting it into the priority queue. (Some priority queues directly support operations like *decrease-key*. We shall not assume such capabilities here.)

We combine this non-meldable priority queue with the union-find data structure to obtain a meldable priority queue. Like the sets in the union-find data structure, each meldable priority queue will have a representative root element identifying the priority queue. This contrasts the non-meldable priority queues that were of a separate type. This root element may not be the one with the smallest key. The meldable priority queues supports the following operations:

MAKE-PQ(x)	– Create and return a priority queue containing the single element x.
INSERT(x, y)	– Insert element y into the priority queue whose root is x.
DELETE(x)	– Delete element x from the priority queue containing it.
FIND-MIN(x)	– Find and return an element with the smallest key in priority queue whose root is x.
MELD(x, y)	– Meld the queues whose root elements are x and y returning the new root.
CHNG-KEY(x, k)	– Change the key associated with element x to k.

The operations *INSERT*(x, y) and *FIND-MIN*(x) assume that x is the root element of its priority queue. Similarly, *MELD*(x, y) assumes that x and y are root elements. It is possible to extend the data structure with an additional union-find data structure that supports a *find*(x) operation that returns the root element of the priority queue containing x. (As shown in Kaplan *et al.* [19], a meldable priority queue data structure that supports a *MELD*(x, y) operation that melds the priority queues containing the elements x and y, where x and y are not necessarily representative elements must include, at least implicitly, an implementation of a union-find data structure.)

A collection of meldable priority queues is now maintained as follows. Each priority queue of the collection is maintained as a tree of a union-find data structure. Each element x contained in such a tree thus has a parent pointer $p[x]$ and a rank $rank[x]$ assigned to it by the union-find data structure. In addition to that, each element x has a 'local' priority queue $PQ[x]$ associated with it. This priority queue contains the element x itself, and the minimal element of each subtree of x. (Thus if x has d children, $PQ[x]$ contains $d+1$ elements.) If x is at the root of a union-find tree, then to find the minimal element in the priority queue of x, a *FIND-MIN*(x) operation, we simply need to find the minimal element is the priority queue $PQ[x]$, a *find-min*$(PQ[x])$ operation.

When an element x is first inserted into a priority queue, by a *MAKE-PQ*(x) operation, we initialize the priority queue $PQ[x]$ of x to contain x, and no other element. We also set $p[x]$ to x, to signify that x is a root, and set $rank[x]$ to 0.

If x and y are root elements of the union-find trees containing them, then a *MELD*(x, y) operation is performed as follows. As in the union-find data structure, we compare the ranks of x and y and hang the element with the smaller rank on the element with the larger rank. If the ranks are equal we decide, arbitrarily, to hang x on y and we increment $rank[y]$. Finally, if x is hung on y, then to maintain the invariant condition stated above, we insert the minimal element in $PQ[x]$ into $PQ[y]$, an *insert*$(PQ[y],$ *find-min*$(PQ[x]))$ operation. (If y is hung on x we perform an analogous *insert*$(PQ[x],$ *find-min*$(PQ[y]))$ operation.)

MAKE-PQ(x) :

$p[x] \leftarrow x$
$rank[x] \leftarrow 0$
$PQ[x] \leftarrow$ *make-pq*(x)

INSERT(x, y) :

MAKE-PQ(y)
MELD(x, y)

DELETE(x) :

CHNG-KEY$(x, +\infty)$

FIND-MIN(x) :

find-min$(PQ[x])$

CHNG-KEY(x, k) :

change-key$(PQ[x], x, k)$
FIND(x)

MELD(x, y) :

if $rank[x] > rank[y]$
  then
    HANG(y, x)
    return x
  else
    HANG(x, y)
    if $rank[x] = rank[y]$
    then
      $rank[y] \leftarrow rank[y]+1$
    return y

FIND(x) :

CUT-PATH(x)
COMPRESS-PATH(x)
return $p[x]$

CUT-PATH(x) :

if $p[x] \neq x$ then
  CUT-PATH$(p[x])$
  UNHANG$(x, p[x])$

COMPRESS-PATH(x) :

if $p[x] \neq x$ then
  COMPRESS-PATH$(p[x])$
  HANG$(x, p[p[x]])$

HANG(x, y) :

insert$(PQ[y],$ *find-min*$(PQ[x]))$
$p[x] \leftarrow y$

UNHANG(x, y) :

delete$(PQ[y],$ *find-min*$(PQ[x]))$

Figure 2: A meldable priority queue obtained by placing a non-meldable priority queue at each node of the union-find data structure.

A *DELETE*(x) operation, which deletes x from the priority queue containing it is implemented in the following indirect way. We change the key associated with x to $+\infty$, using a *CHNG-KEY*$(x, +\infty)$ operation, to signify that x was deleted, and we make the necessary changes to the data structure, as described below. Each priority queue in our collection keeps track of the total number of elements contained in it, and the number of deleted elements contained in it. When the fraction of deleted elements exceeds a half, we simply rebuild this priority queue. A standard argument shows that this rebuilding affects the amortized cost of all the operations by only a constant factor.

How do we implement a *CHNG-KEY*(x, k) operation then? If x is a root element, we simply change the key of x in $PQ[x]$ using a *change-key*$(PQ[x], x, k)$ operation. If x is not a root, then before changing the key of x we perform a *FIND*(x) operation. A *FIND*(x) operation compresses the path connecting x to the root by cutting all the edges along the path and hanging all the elements encountered directly on the root. Let $x = x_1, x_2, \ldots, x_k$ be the sequence of elements on the path from x to the root of its tree. For $i = k-1, k-2, \ldots, 1$ we *unhang* x_i from x_{i+1}. This is done by removing *find-min*$(PQ[x_i])$ from $PQ[x_{i+1}]$. After that, we hang all the elements $x_1, x_2, \ldots, x_{k-1}$ on x_k. This is done by setting $p[x_i]$ to x_k and by adding *find-min*$(PQ[x_i])$ to $PQ[x_k]$. (Note that we also unhang x_{k-1} from x_k and then hang it back.)

If x is not a root element then after a *FIND*(x) operation, x is a child of the root. Changing the key of x is now relatively simple. We again unhang x from $p[x]$, change the key of x and then hang x again on $p[x]$. A moment's reflection shows that it is, in fact, enough just to change the key of x in $PQ[x]$, and then perform a *FIND*(x) operation. The element x may temporarily be contained in some priority queues with a wrong key, but this will immediately be corrected.

A simple implementation of all these operations is given in Figure 2. The important thing to note is that the operation of a meldable priority queue mimics the operation of a union-find data structure and that changing a pointer $p[x]$ from y to y' is accompanied by calls to *UNHANG*(x, y) and *HANG*(x, y').

Simple analysis Since the union-find data structure makes only an amortized number of $O(\alpha(n, n))$ hangings and unhangings per union or find operation, we immediately get that each meldable priority queue operation takes only $O(pq(n)\,\alpha(n, n))$ amortized time. This is precisely the analysis given by van Emde Boas *et al.* [36]. In the next

section, we tighten the analysis so as to get *no* asymptotic overhead with existing priority queues.

4 The improved analysis

In this section we present our main contribution, an improved analysis of the transformation of van Emde Boas *et al.* [36] described in the previous section. We assume that the non-meldable priority queue $\mathcal{P}$ supports *insert*, *delete* and *find-min* operations in $O(pq(n))$ (expected) amortized time. By a simple transformation of Alstrup *et al.* [1], we can actually assume that the amortized cost of *insert* and *find-min* operations is $O(1)$ and that the amortized cost of *delete* operations is $O(pq(n))$. We now claim:

Theorem 4.1 *If $\mathcal{P}$ is a priority queue data structure that supports insert and find-min operations in $O(1)$ (expected) amortized time and delete operations in $pq(n) = O(\log n)$ (expected) amortized time, then $\mathcal{T}(\mathcal{P})$ is a priority queue data structure that supports insert, find-min and meld operations in $O(1)$ (expected) amortized time and delete operations in $O(pq(n)\,\alpha(n, n/pq(n)))$ (expected) amortized time, where $\alpha(m,n)$ is the inverse Ackermann function appearing in the analysis of the union-find data structure, and n is the total number of operations performed on all the priority queues.*

Proof: Consider a sequence of n operations on the data structure, of which $f \leq n$ are *DELETE* or *CHNG-KEY* operations. (Each such operation initiates a *FIND* operation, hence the choice of the letter f.) Our aim is to show that the cost of carrying out all these operations is $O(n + f\,pq(n)\,\alpha(n, n/pq(n)))$. This bounds the amortized cost of each operation in terms of the maximum number of elements contained in all the priority queues. (See a discussion of this point at the end of the section.)

All the operations on the data structure are associated with changes made to the parent pointers $p[x]$ of the elements contained in the priority queues. To change the parent $p[x]$ of x from y to y', we first call $UNHANG(x, y)$ which performs a delete operation on $PQ[y]$, and then call $HANG(x, y')$ which performs an insert operation on $PQ[y']$ and sets $p[x]$ to y'. As insert operations are assumed to take constant time, we can focus on the delete, or *UNHANG*, operations. As the total number of pointer changes made in the union-find data structure is at most $O(n\,\alpha(n,n))$, and as each priority queue acted upon is of size at most n, we get immediately an upper bound of $O(n\,pq(n)\,\alpha(n,n))$ on the total number of operations performed. We want to do better than that.

If element x is a root of one of the union-find trees, we let $size(x)$ be the number of elements contained in its tree. If x is no longer a root, we let $size(x)$ be the number of descendants it had just before it was hanged on another element. It is easy to prove by induction that we always have $size(x) \geq 2^{rank[x]}$: The statement is trivial if $rank[x] = 0$, and otherwise, x got its rank by the union of two roots of rank $rank[x] - 1$.

Let

$$p = pq(n) + \frac{n}{f} \quad , \quad S = p^2 \quad , \quad L = \log S \; .$$

We say that an element x is *big* if $size(x) \geq S$. Otherwise, it is said to be *small*. We say that an element x is *high* if $rank[x] \geq L$. Otherwise, it is said to be *low*. Note that if an element is big (or high), so are all its ancestors. We also note that all high elements are big, but big elements are not necessarily high. We let *SMALL*, *BIG*, *LOW* and *HIGH* be the sets of small, big, low and high vertices, respectively. As noted above, we have $SMALL \subseteq LOW$ and $HIGH \subseteq BIG$ but $LOW \cap BIG$ may be non-empty.

Below we bound the total cost of all the $UNHANG(x, p[x])$ operations. All other operations take only $O(n)$ time. We separate the analysis into the following five cases:

Case 1: $x, p[x] \in SMALL$

We are doing at most f path compressions. Each path in the union-find forest contains at most L small elements. (This follows from the invariant $rank[p[x]] > rank[x]$ and from the fact that high elements are big.) Thus, each path compression involves at most L unhang operations in which $x, p[x] \in SMALL$. As each priority queue involved is of size at most S, the total cost is

$$O(f \cdot L \cdot pq(S)) \;=\; O(f \cdot p) \;=\; O(n + f \cdot pq(n)) \; .$$

(Note that $L = \log S = O(\log p)$ and that $pq(S) = O(\log S) = O(\log p)$. (We assume that $pq(n) = O(\log n)$.) Hence $L \cdot pq(S) = O(\log^2 p) = O(p)$.)

Case 2: $x \in SMALL$ **and** $p[x] \in BIG$.

In each one of the f path compressions performed there is at most one unhang operation of this form. (As ancestors of big elements are also big.) Hence, the total cost here is $O(f\, pq(n))$.

Case 3: $x, p[x] \in BIG \cap LOW$.

To bound the total cost of these operations we bound the number of elements that are contained at some stage in $BIG \cap LOW$. An element is said to be a *minimally-big* element if it is big but all its descendants are small. As each element can have at most one minimally-big ancestor, and each minimally-big element has at least S descendants, it follows that there are at most n/S minimally-big elements. As each big element is an ancestor of a minimally-big element, it follows that there are at most Ln/S elements in $BIG \cap LOW$.

An element $x \in BIG \cap LOW$ can be unhanged from at most L other elements of $BIG \cap LOW$. (After each such operation $rank[p[x]]$ increases, so after at most L such operations $p[x]$ must be high.) The total number of operations of this form is at most $L^2 n/S < n/p$. Thus, the total cost of all these operations is $O(n\, pq(n)/p) = O(n)$.

Case 4: $x \in BIG \cap LOW$ **and** $p[x] \in HIGH$.

As in Case 2, each one of the f path compressions performed causes at most one unhang operation of this form. (As ancestors of high elements are also high.) Hence, the total cost here is $O(f\, pq(n))$.

Case 5: $x, p[x] \in HIGH$.

To bound the number of $UNHANG(x, p[x])$ operations in which $x, p[x] \in HIGH$, we rely on Lemma 2.2. As each $UNHANG(x, p[x])$ operation, where $x \in HIGH$ is associated with a parent pointer change of a high vertex, it follows that the total number of such operations is at most $O(\frac{n}{S} + f\,\alpha(f + \frac{n}{S}, \frac{n}{S})) = O(f\,\alpha(f, \frac{n}{S}))$. (Note that $S = 2^L$ and $f \geq n/S$.) Now

$$\alpha(f, \frac{n}{S}) \;\leq\; \alpha(\frac{n}{p}, \frac{n}{p^2}) \;\leq\; \alpha(n, \frac{n}{p}) \;\leq\; \alpha(n, \frac{n}{pq(n)})\,.$$

This chain of inequalities follows from the fact that $f \geq n/p$ and from simple properties of the $\alpha(m, n)$ function. (The $\alpha(m, n)$ function is decreasing in its first argument, increasing in the second, and $\alpha(m, n) \leq \alpha(cm, cn)$, for $c \geq 1$.) As the cost of each *delete* operation is $O(pq(n))$, the cost of all unhang operations with $x, p[x] \in HIGH$ is at most $O(f \cdot pq(n) \cdot \alpha(n, n/pq(n)))$.

The total cost of all unhang operations is therefore $O(n + f\, pq(n)\, \alpha(n, n/pq(n)))$, as required. □

5 Using atomic heaps

Atomic heaps, invented by Fredman and Willard [15], are non-meldable priority queues that support *insert*, *find-min* and *delete* operations in *constant* amortized time, when the number of elements in the priority queue is $O(\log^2 n)$. The constant amortized operation time is obtained by using a carefully prepared look-up table of size $O(n)$ that can be constructed in $O(n)$ time. The same table can be used to maintain many small priority queues. Atomic heaps support some additional operations, like predecessor search, but these are not needed here.

We can use atomic heaps to obtain a slightly improved version of the transformation from non-meldable priority queues to meldable ones. The modified transformation, which we denote by $\mathcal{T}^A$, is almost identical to the transformation of Section 3. The only difference is that atomic heaps are used at nodes of the union-find data structure that are of small size. As in the previous section, we define $size(x)$ as follows: if x is a root then $size(x)$ is the number of descendants of x, and if x is not a root, then $size(x)$ is the number of descendants that x had just before it was hung on another node. Given a threshold S, we say that a node x is *small*, is $size(x) \leq S$, and *big*, otherwise. As deleted elements are only marked as deleted, until the whole priority queue is rebuilt, $size(x)$ never decreases.

Let n be a bound on the total number of operations that will be performed on the collection of priority queues that we are supposed to maintain. Suppose, at first, that we know n in advance. (We will remove this assumption shortly.) Let $\mathcal{P}$ be the non-meldable priority queue data structure supplied to the transformation. We refer to priority queues

maintained using this data structure as $\mathcal{P}$-heaps. Let $S = \log^2 n$. If x is a small node in the union-find forest, i.e., if $size(x) \leq S$, then instead of placing in x a $\mathcal{P}$-heap, we place in x an atomic heap. When a small node x becomes big, the elements in the atomic heap of x are moved into a newly created $\mathcal{P}$-heap. This transition is paid for by the original insertion of the elements into the atomic heap. As $size(x)$ never decreases, a big node never becomes small, so a $\mathcal{P}$-heap is never converted back into an atomic heap.

If n, the total number of operations to be performed, is not known in advance, we can use a simple doubling technique. We let n be the number of operations performed so far. When the number of operations doubles, we rebuild all the priority queues. It is easy to see that this changes the amortized cost of each operation by only a constant factor. We now claim:

Theorem 5.1 *If $\mathcal{P}$ is a priority queue data structure that supports insert and find-min operations in $O(1)$ (expected) amortized time and delete operations in $pq(n) = O(\log n)$ (expected) amortized time, then $\mathcal{T}^A(\mathcal{P})$ is a priority queue data structure that supports insert, find-min and meld operations in $O(1)$ (expected) amortized time and delete operations in $O(pq(n) + \alpha(n,n))$ (expected) amortized time, where $\alpha(m,n)$ is the inverse Ackermann function appearing in the analysis of the union-find data structure, and n here is the total number of operations performed on all the priority queues.*

Proof: The proof is very similar to the proof of Theorem 4.1. Our goal is to show that the total cost of all the unhang operations is bounded by $O(n + f(pq(n) + \alpha(n,n)))$, where f is the number of *delete* operations performed. We use the same definitions of the sets *SMALL*, *BIG*, *LOW* and *HIGH* used in the proof of Theorem 4.1, but with the following modified parameters:

$$p = \log n \quad , \quad S = p^2 = \log^2 n \quad , \quad L = \log S = 2 \log\log n \; .$$

We break the analysis into the same five cases:

Case 1: $x, p[x] \in SMALL$

As the cost of each operation on an atomic heap requires only $O(1)$ time, the total cost of all the operations here is $O(n + f\,\alpha(n,n))$.

Case 2: $x \in SMALL$ **and** $p[x] \in BIG$.

In each one of the f path compressions performed there is at most one unhang operation of this form. (As ancestors of big elements are also big.) Hence, the total cost here is $O(f\,pq(n))$.

Case 3: $x, p[x] \in BIG \cap LOW$.

The total number of unhang operations of this form is at most $L^2 n/S < n/p = n/\log n$. (See the proof of Theorem 4.1.) Thus, the total cost of all these operations is $O(n\,pq(n)/\log n) = O(n)$.

Case 4: $x \in BIG \cap LOW$ **and** $p[x] \in HIGH$.

As in Case 2, each one of the f path compressions performed causes at most one unhang operation of this form. (As ancestors of high elements are also high.) Hence, the total cost here is $O(f\,pq(n))$.

Case 5: $x, p[x] \in HIGH$.

By Lemma 2.2, the number of unhang operations of this form is at most $O(\frac{n}{S} + f\,\alpha(f + \frac{n}{S}, \frac{n}{S}))$. The cost of each such operation is at most $O(pq(n))$. If $f > n/(\log n)^{3/2}$, then $\alpha(f + \frac{n}{S}, \frac{n}{S})) = O(1)$, and the total cost is $O((\frac{n}{S} + f)pq(n)) = O((\frac{n}{\log^2 n} + f)pq(n)) = O(n + f\,pq(n))$, as required. If $f \leq n/(\log n)^{3/2}$ then the total cost is $O(\frac{n}{(\log n)^{3/2}}\, pq(n)\, \alpha(n,n)) = O(n)$.

The total cost of all unhang operations is therefore $O(n + f(pq(n) + \alpha(n,n)))$, as required. □

6 Bounds in terms of the maximal key value

In this section we describe a simple transformation, independent of the transformation of Section 3, that speeds up the operation of a meldable priority queue data structure when the keys of the elements are integers taken from the

range $[1, N]$, where N is small relative to n, the number of elements. More specifically, we show that if $\mathcal{P}$ is a meldable priority queue data structure that supports *delete* operations in $O(pq(n))$ amortized time, and all other operations in $O(1)$ amortized time, where n is the number of elements in the priority queue, then it is possible to transform it into a meldable priority queue data structure $\mathcal{T}'(\mathcal{P})$ that supports *delete* operations in $O(pq(\min\{n, N\}))$ amortized time, and all other operations in $O(1)$ time. To implement this transformation we need random access capabilities, so it cannot be implemented on a pointer machine.

To simplify the presentation of the transformation, we assume, at first, that a *delete* operation receives references to the element x to be deleted *and* to the priority queue containing it. This is a fairly standard assumption.[1] Note, however, that the *delete* operation obtained by our first transformation is stronger as it only requires a reference to the element, and not to the priority queue. We later show how to dispense with this assumption.

The new data structure $\mathcal{T}'(\mathcal{P})$ uses two different representations of priority queues. The first representation, called the *original*, or *non-compressed* representation is simply the representation used by $\mathcal{P}$. The second representation, called the *compressed* representation, is composed of an array of size N containing for each integer $k \in [1, N]$ a pointer to a doubly linked list of the elements with key k contained in the priority queue. (Some of the lists may, of course, be empty.) In addition to that, the compressed representation uses an original representation of a priority queue that holds the up to N distinct keys belonging to the elements of the priority queue.

Initially, all priority queues are held using the original representation. When, as a result of an *insert* or a *meld* operation, a priority queue contains more than N elements, we convert it to compressed representation. This can be easily carried out in $O(N)$ time. When, as a result of a *delete* operation, the size of a priority queue drops below $N/2$, we revert back to the original representation. This again takes $O(N)$ time. The original representation is therefore used to maintain *small* priority queues, i.e., priority queues containing up to N elements. The compressed representation is used to represent *large* priority queues, i.e., priority queues containing at least $N/2$ elements. (Priority queues containing between $N/2$ and N elements are both small and large.)

By assumption, we can insert elements to non-compressed priority queues in $O(1)$ amortized time, and delete elements from then in $O(pq(n)) = O(pq(N))$ amortized time. We can also insert an element into a compressed priority queue in $O(1)$ amortized time. We simply add the element into the appropriate linked list, and if the added element is the first element of the list, we also add the key of the element to the priority queue. Similarly, we can delete an element from a compressed priority queue in $O(pq(N))$ amortized time. We delete the element from the corresponding linked list. If that list is now empty, we delete the key from the non-compressed priority queue. As the compressed priority queue contains at most N keys, that can be done in $O(pq(N))$ amortized time. Since *insert* and *delete* operations are supplied with a reference to the priority queue to which an element should be inserted, or from which it should be deleted, we can keep a count of the number of elements contained in the priority queue. This can be done for both representations. (Here is where we use the assumption made earlier. As mentioned, we will explain later why this assumption is not really necessary.) These counts tell us when the representation of a priority queue should be changed.

A small priority queue and a large priority queue can be melded simply be inserting each element of the small priority queue into the large one. Even though this takes $O(n)$ time, where n is the number of elements in the small priority queue, we show below that the amortized cost of this operation is only $O(1)$.

Two large priority queues can be easily melded in $O(N)$ time. We simply concatenate the corresponding linked lists and add the keys that are found, say, in the second priority queue, but not in the first, into the priority queue that holds the keys of the first priority queue. The second priority queue is then destroyed. We also update the size of the obtained queue. Again, we show below that the amortized cost of this is only $O(1)$.

Theorem 6.1 *If $\mathcal{P}$ is a priority queue data structure that supports insert, find-min and meld operations in $O(1)$ (expected) amortized time and delete operations in $O(pq(n))$ (expected) amortized time, then $\mathcal{T}'(\mathcal{P})$ is a priority queue data structure that supports insert, find-min and meld operations in $O(1)$ (expected) amortized time and delete operations in $O(pq(\min\{n, N\}))$ (expected) amortized time. Here n is the total number of operations performed on all the priority queues, and N is the maximal key value.*

[1] A reference to the appropriate priority queue can be obtained using a separate union-find data structure. The amortized cost of finding a reference is then $O(\alpha(n, n))$. This is *not* good enough for us here as we are after bounds that are independent of n.

Proof: We use a simple potential based argument. The potential of a priority queue held in original, non-compressed, representation is defined to be $1.5n$, where n the number of elements contained in it. The potential of a compressed priority queue is N, no matter how many elements it contain. The potential of the whole data structure is the sum of the potentials of all the priority queues.

The operations *insert*, *delete* and *find-min* have a constant actual cost and they change the potential of the data structure by at most an additive constant. Thus, their amortized cost is constant.

Compressing a priority queue containing $N \leq n \leq 2N$ elements requires $O(N)$ operations but it reduces the potential of the priority queue from $1.5n$ to N, a drop of at least $N/2$, so with proper scaling the amortized cost of this operation may be taken to be 0. Similarly, when a compressed priority queue containing $n \leq N/2$ elements is converted to original representation, the potential of the priority queue drops from N to $1.5n$, a drop of at least $N/4$, so the amortized cost of this operation is again 0.

Melding two original priority queues has a constant actual cost. As the potential of the data structure does not change, the amortized cost is also constant. Melding two compressed priority queues has an actual cost of $O(N)$, but the potential of the data structure is decreased by N, so the amortized cost of such meld operations is again 0. Finally, merging a small priority queue of size $n \leq N$, in original representation, and a compressed priority queue has an actual cost of $O(n)$ but the potential decreases by $1.5n$, giving again an amortized cost of 0. This completes the proof. □

We next explain the small modification needed in the transformation described above to accommodate *delete* operations that only get a reference to the element to be deleted. The problem is that without a reference to the priority queue acted upon, *delete* operations cannot decrement, in constant time, the counter holding the number of elements contained in the priority queue. Still, *insert* operations can increment such a counter as they do get a reference to the priority queue. We can thus maintain, for each non-compressed priority queue, a counter *ins* that counts the number of insertions made into it. When two priority queues are melded, we add these counters. We convert a priority queue into compressed representation when its counter *ins* exceeds N. The potential of a non-compressed priority queue is now defined to be $1.5\,ins$.

We have no way of knowing when the actual size of priority queue drops below $N/2$. The only reason, however, for converting compressed priority queues back into non-compressed representation is to save space. We can, however, reclaim space on a global rather than local basis. Instead of maintaining the individual size of each compressed priority queue, we maintain the number b of compressed priority queues, and their total size s. When the total size s drops below $bN/4$, i.e., when the space utilization drops below $1/4$, we naively compute the size of each compressed priority queue, and convert it to a non-compressed representation if its size is below $N/2$. (The *ins* counter of such a priority queue is then set to its actual size.) This can be easily done in $O(bN)$ time. As at most $b/2$ of the priority queues can have more than $N/2$ elements, this reduces the potential of the whole data structure by at least $b/2 \cdot (N/4)$, so the amortized cost of this 'garbage collection' is 0. This completes the description of the required modification.

7 Concluding remarks

We presented an improved analysis of a general transformation, first presented in van Emde Boas [36], that adds a meld operation to priority queue data structures that do not support it, with essentially *no extra cost*. (The analysis of [36] only showed that the extra cost is tiny.) We also presented a second transformation that speeds up the operations of meldable priority queues when the range of the possible keys is small.

Combined with Thorup's [32] technique of transforming sorting algorithms into priority queue data structures, our result can be stated as follows: A sorting algorithm that sorts up to n elements in $O(n\,s(n))$ time, where $s(n) = \Omega(\alpha(n,n))$, can be converted into a *meldable* priority queue data structure that supports delete operations in $O(s(n))$ amortized time, and all other operations in $O(1)$ amortized time.

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